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Abstract

We address the question of the measurement of pure health inequalities and achievement in the context of welfare decreasing variables. We adopt a general framework whereby the health variable is reported on an interval, from an optimum level m to a critical survival threshold b. There are two problems that require some departures from the usual framework used to measure inequality and social welfare. Firstly, we show that for welfare decreasing variables, the equally distributed equivalent value is decreasing in progressive transfers (instead of being increasing). Accordingly, appropriate achievement and inequality indices for welfare decreasing variables are introduced. Secondly, because the Lorenz curve and the associated inequality indices are not robust to alternative values of the survival threshold, we argue that the family of translation invariant social welfare functions and related absolute Lorenz curve allow us to undertake inequality comparisons between distributions that are robust to the chosen level of the survival threshold. An illustrative application of the methodology is provided.

Keywords: Health achievement and inequality; welfare decreasing variables; survival thresholds; relative and absolute Lorenz curves.

JEL codes: D63, I14

1 Introduction

The improvement of key health indicators has been a major concern of the development debate for many decades, and remains so today, as formulated for instance in the MDGs (2000-2015) and SDGs (2015-2030). Beyond improving the average value of key indicators, it has increasingly been recognized that the shape of the distribution is also in need of attention. There are several reasons for turning our attention to inequality in the distribution of a health variable. In the case of calorie intake for instance, low levels of nutrition are associated with stunting in infants and certain severe deficiencies for adults (typically, iron, vitamin A and iodine deficiency). High levels of energy intake are also problematic, as they increase the risk of cardiovascular disease and type II diabetes (WHO, 2011).

Additionally, there are the usual normative concerns for preferring less to more inequality in health, in relation to two distributions with the same mean value. Thus, greater emphasis and interest by researchers in the last twenty years has placed the measurement of socio-economic inequality and achievement in health at the center of the development debate (Wagstaff et al 2003, Wagstaff 2002, Erreygers 2013), as opposed to simply improving the aggregate indicators such as life expectancy and the reduction in infant mortality rates. But there are equally important contexts where the focus is on pure rather than socio-economic inequalities in health (e.g. Osmani, 1992; Sen, 2002; Bommier and Stecklov, 2002). The measurement of inequalities in the context of self-assessed health (Allison and Foster, 2004; Apouey, 2007; Abul Naga and Yalcin, 2008, Arrighi et al., 2011; Kobus and Milos, 2012) is also concerned with quantifying pure health inequalities.

One problem with subjective self-assessed health (SAH) data, however, is that they have been shown to be biased particularly in the context of developing countries, in that they entail a reverse gradient between health and socio-economic status (van Doorslaer and O'Donnell, 2011). The research context of our paper is, therefore, the measurement of pure health inequalities and achievement in relation to objective measure of health. Specifically, we examine the context of welfare decreasing health variables in relation to objective health indicators such as sugar level, cholesterol intake, body mass, that exhibit an inverted U relation with health status. Other variables of course exist for which it is accepted that any level of consumption does not improve health and may harm if consumed in significant amounts. Such variables include lead contamination, nicotine intake, dioxin etc. We accommodate the first set of variables by measuring inequality when the health variable is reported on an interval(m, b), from an optimum level m to a critical level

b, beyond which survival is no longer likely. For instance, in the context of sugar level, the lower bound for survival is 40 milligrams of glucose per deciliter of blood, and the corresponding critical upper bound is $b = 450$ milligrams per deciliter. In the context of body mass the lower bound is generally taken to be $a = 10$ kilograms per squared meter, b is approximately equal to 60, while m can generally be any values chosen in the interval of 18.5 to 24.9 (WHO, 2004; 2017).

In the context of the second type of variables, our analysis equally applies by setting the optimum level m to zero. The emphasis of this paper is on the upper tail of the health indicator, the interval from m to b, as the measurement of inequality and achievement is generally well understood in the context of welfare increasing variables (income being the leading example of course). Furthermore, there are interesting contributions in the context of poverty measurement in relation to resource variables that exhibit an inverted U relation with well-being (for instance Apablaza et al, 2016).

The context of inequality measurement *per se* on a welfare decreasing variable should not be seen as problematic: the Hardy, Littlewood and Polya theorems (Hardy et al, 1952) relate the class of Schur convex functions to progressive transfers that are applied to a distribution of interest. The fact that the utility function is decreasing in a particular health indicator does not invalidate the use of the Lorenz curve or entropy type indices in ranking health distributions: what matters is the Schur-convexity of the inequality measure, or the Schur-concavity of the underlying social welfare function.

There are nonetheless two unresolved problems that require attention. Firstly, we note that a large family of inequality indices are derived in association with a social welfare function. The inequality index is derived via a comparison of the mean of a variable with the equally distributed value of the distribution: this is the so called Atkinson-Kolm-Sen approach which measures the level of equality as a ratio of the equally distributed equivalent value to the mean of the distribution. Health achievement indices (Wagstaff, 2002) are derived also as the equally distributed equivalent (alternatively, the mean scaled by the level of equality in the distribution.) We show however in the paper that for welfare decreasing variables, the equally distributed equivalent value is a Schur-convex function: that is, the equally distributed equivalent value is decreasing in progressive transfers. This is the opposite of what we should expect of such a summary statistic. In particular, a naive computation of (say) an Atkinson (1970) inequality index on a distribution exhibiting some positive level of dispersion will always entail that inequality is smaller than zero.

The second problem that requires attention is that of survival thresholds. When measuring health inequality and achievement, we are concerned with deviations of the individual observations from the critical threshold b. Clinical research can of course inform about sensible values of b. Nonetheless, it remains that the Lorenz curve and scale invariant inequality indices will take different values for different choices of the survival threshold. As it turns out, this second problem in fact reignites the debate regarding rightist versus leftist inequality and social welfare indices (Kolm, 1976 a,b). While we do not propose to take sides in this debate, we note that in relation to translation invariant social welfare functions it is possible to derive health inequality and achievement indices that are robust to the choice of survival thresholds. In the same way, the absolute Lorenz curve (Moyes, 1987) allows us to achieve inequality comparisons between distributions that are robust to the chosen level of the survival threshold, whereas the classical Lorenz curve fails to be invariant to the choice of the parameter b. The class of welfare decreasing translation invariant social welfare functions and related inequality and achievement indices thus provide an attractive solution in the context of our research question. For the sake of completeness, and in order to provide a readily applicable methodology, we also provide in sections 2 and 3 of the paper axiomatically derived functions for achievement and inequality indices in the context of welfare decreasing variables, measured in deviation from survival thresholds. Definitions of the underlying relative, generalized, and absolute Lorenz curves are also provided. Section 4 of the paper illustrates briefly our methodology in the context of health achievement and inequality comparisons in a group of five Arab countries. Section 5 concludes the paper.

2 Welfare decreasing variables and the measurement of health achievement and inequality

In the Atkinson-Kolm-Sen approach, the derivation of inequality indices is approached in relation to a family of social welfare function taken to capture society's preferences for greater health achievement, and less health inequality. The inequality index is derived via a comparison of the mean of a variable with the equally distributed value of the distribution. Health achievement indices (Wagstaff, 2002) are derived also as the equally distributed equivalent (alternatively, the mean scaled by the level of equality in the distribution.) One purpose of this section is to show that for welfare decreasing variables, the equally distributed equivalent value is decreasing in progressive transfers (when we would expect the opposite from such a summary statistic). We then dwell further on the implications of this finding for alternative specifications of achievement and AKS inequality indices for welfare decreasing health variables.

2.1 Fundamental axioms

Let $W : [m, b)^n \rightarrow \mathbb{R}$ denote a social welfare function in relation to a welfare decreasing health variable. We measure welfare with reference to individuals' position from the upper survival threshold b . We let ι_n denote an n-dimensional vector of ones, $\iota_n = (1, \ldots, 1)$, and we consider several axioms commonly used for social welfare functions [1]. In what follows therefore $b\iota_n - Y$ is a compact notation for the vector $(b - y_1, \cdots, b - y_n)$. We begin by stating three standard properties ADD, $ANON$ and REP. The first of these, ADD, captures the notion that the social welfare function is the average of welfare levels experienced by individuals. $ANON$ is an anonymity axiom that insures that only the endowment levels y_i matter for the measurement of social welfare. REP is an axiom of invariance of the social welfare function to certain types of population replications.

- ADD (Strong independence principle) The social welfare function is additively separable in the utility functions of the n individuals.

- $ANON$ (Anonymity) For any $n \times n$ permutation matrix Π and any distribution $Y \in [m, b)^n$,there holds $W\left((b\iota_n - Y)\Pi; m\right) = W\left(b\iota_n - Y; m\right)$.

- REP (Invariance to population replication) For any distribution $Y \in [m, b)^n$, replication of the vector Y to a new distribution $(Y, Y) \in [m, b)^{2n}$ leaves social welfare unchanged: $W\left(b\iota_n - Y; m\right) = W(b\iota_{2n} - (Y, Y); m)$.

Our next two axioms formalize the effect of certain transformations of the distribution Y on social welfare. The monotonicity axiom MON requires that social welfare increases when individual endowments y_i are reduced. Preference for greater equality is introduced via the axiom $EQUAL$, requiring that social welfare increases with Pigou-Dalton transfers.

- MON (Monotonicity) The social welfare function W is strictly decreasing $y_1, \ldots, y_n$.

- $EQUAL$ (Social aversion to inequality): $W\left(b\iota_n - Y; m\right)$ is strictly increasing in Pigou-Dalton transfers.

[1] For a detailed discussion of these axioms, see for instance Kolm (1976 a,b) or Champernowne and Cowell (1998).

Finally we discuss two invariance axioms capturing certain transformations of the data that leave the ordering of distributions by the social welfare function unchanged. The first of these, $SCALINV$, guarantees that social welfare does not change when units of measurement are modified in a particular manner. The second, $TRANSINV$, is used to capture the notion that inequality is invariant to translation shifts of the distribution of resources.

- $SCALINV$ (Scale invariance) For any scalar $\lambda > 0$, and for any pair of distributions $X, Y \in [m, b)^n$, $W(b\iota_n - Y; m) \geq W(b\iota_n - X; m) \Longleftrightarrow W(\lambda b\iota_n - \lambda Y; \lambda m) \geq W(\lambda b\iota_n - \lambda X; \lambda m)$

- $TRANSINV$ (Translation invariance) For any admissible value λ and for any pair of distributions $X, Y \in [m, b)^n$, $W(b\iota_n - Y; m) \geq W(b\iota_n - X; m) \Longleftrightarrow W(b\iota_n - (Y + \lambda\iota_n); m) \geq W(b\iota_n - (X + \lambda\iota_n); m)$.

It is a well known result (see for instance Kolm, 1976a) that together the first three axioms entail a social welfare function of the form

$$W(b - y_1, \cdots, b - y_n; m) := \frac{1}{n}\sum_{i=1}^{n}\phi(b - y_i)$$

The monotonicity axiom MON restricts the derivative of the function ϕ to be decreasing on $[m, b)$. On the other hand, the social welfare function satisfies the social aversion to inequality axiom, $EQUAL$, if ϕ is concave on $[m, b)$. Equivalently, following Apablaza et al. (2016), there are two distinct elementary transformations of the distribution Y that contribute to improving social welfare: (i) a Pigou-Dalton transfer on $[m, b)^n$, and (ii) a decrement (reduction) of some y_i on $[m, b)^n$. Formally, let $\delta > 0$, and consider two observations y_l and y_k such that $y_l - \delta \geq y_k + \delta$. A new distribution $Y^* = (y_1^*, \ldots, y_n^*) \in [m, b)^n$ is obtained from Y via a Pigou-Dalton transfer if $y_l^* = y_l - \delta$, $y_k^* = y_k + \delta$ and $y_i^* = y_i$ for all $i \neq l, k$. The distribution $Y^* = (y_1^*, \ldots, y_n^*) \in [m, b)^n$ is obtained from Y via a unique decrement if for some $\delta \leq y_j - m$, $y_j^* = y_j - \delta$ and $y_i^* = y_i$ for all $i \neq j$. Observe that decrements modify the sum total of a distribution, whereas Pigou-Dalton transfers preserve the mean (and sum total) of the original distribution.

Together the axioms ADD, $ANON$, REP, MON, $EQUAL$ and $SCALINV$ restrict the choice of $\phi(b - y_i)$ to the family $v_\beta(b - y)$ of power functions:

$$v_\beta(b - y) = \begin{cases} \dfrac{(b - y)^{1-\beta}}{1 - \beta}, & \beta \geq 0, \quad \beta \neq 1 \\ \ln(b - y), & \beta = 1 \end{cases} \tag{1}$$

Accordingly, the family of social welfare functions that satisfies the above six properties is of the form

$$W_\beta(b - y_1, \cdots, b - y_n; m) = \frac{1}{n}\sum_{i=1}^{n} v_\beta(b - y_i) \tag{2}$$

We shall return to our final axiom, $TRANSINV$ in the next section of the paper.

2.2 Properties of the relative achievement index

Let $\hat{y} \in [m, b)$ denote the equally distributed health level in the distribution Y such that welfare is identical to the level of attainment in the current distribution Y . We have that $v(b - \hat{y}) = W(b - y_1, \cdots, b - y_n; m)$. Following Wagstaff (2002), this equally distributed equivalent value is known as the *achievement index* in the field of health economics. In the income inequality literature, the equally distributed equivalent income is increasing in Pigou-Dalton progressive transfers. The context of welfare decreasing health variables produces a subtle difference:

Proposition 1 *Let $u(.)$ denote any strictly decreasing, and concave function that is differentiable on some closed interval $[m^o, b^o] \subseteq [m, b)$. Then, for any distribution $Y \in [m, b)^n$, with mean $\bar{y}$, the equally distributed equivalent value*

$$\hat{y} := u^{-1}\left(\frac{1}{n}\sum_{i=1}^{n} u(b - y_i)\right) ,$$

is decreasing in Pigou Dalton transfers and satisfies the inequality.

$$\bar{y} \le \hat{y} \tag{3}$$

In the context of equation 5 and 6, the equally distributed equivalent value is in the form of:

$$\hat{y}_R = g(Y; m, b) = \begin{cases} b - \left(\frac{1}{n}\sum\limits_{j=1}^{n} (b - y_j)^{1-\beta}\right)^{1/(1-\beta)} , & \beta > 0, \quad \beta \neq 1 \\ b - \exp\dfrac{1}{n}\sum\limits_{j=1}^{n} \ln(b - y_j), & \beta = 1 \end{cases} \tag{4}$$

The subscript R in $\hat{y}_R$ is introduced to denote achievement indices that satisfy the scale invariance axiom $SCALINV$. The associated AKS inequality indices are referred to as *relative* indices in the income inequality literature, and we shall use this convention here also to distinguish the equally distributed equivalent (4) from the equally distributed equivalent that satisfies the translation invariance axiom $TRANSINV$, that will be denoted as $\hat{y}_A$ in Section 3 below.

2.3 The AKS family of indices and the relative Lorenz curve

It follows from Proposition 1 that application of the standard AKS inequality index introduced by Atkinson (1970), namely the function $I_R(Y) := 1 - (\hat{y}_R/\bar{y})$ will provide the data user with multiple challenges. Firstly, in the light of the inequality (3) the index will usually take on negative values. Secondly, because the index does not depend on the upper threshold b, changes in the units of measurement of y, m and b will change the value taken by the inequality index. More importantly, Pigou-Dalton transfers will increase the value of I_R, suggesting that inequality has increased. It is thus important to adapt the AKS index in the context of welfare decreasing health data, so as to achieve these three desired properties (non-negative property, scale invariance and transfer sensitivity). Consider in particular the following form:

$$\Delta_R(b\iota_n - Y; m) := 1 - \left(\frac{b - \hat{y}_R}{b - \bar{y}}\right) \tag{5}$$

Because $\hat{y}_R \geq \bar{y}$ in the context of welfare decreasing variables (Proposition 1), Δ_R will be a non-negative function. Likewise, Δ_R is now an increasing function of $\hat{y}_R$, as the equally distributed value is a Schur-convex function, decreasing in progressive transfers. Finally, it is clear that the inequality index (5) is invariant to rescaling b, m and the distribution Y by the same constant $\lambda > 0$.

In order to derive a new expression for the relative Lorenz curve that is consistent with our framework, it is useful to consider two ordered vectors associated with the distribution Y : firstly the decreasing rearrangement of Y that we denote by the vector $Y \downarrow = (y_{[1]}, \ldots y_{[n]})$, and secondly the increasing rearrangement of Y that we denote by $Y \uparrow = (y_{(1)}, \ldots y_{(n)})$. Clearly, if we want to maintain the well established and meaningful practice of summing resources starting from the worst off individuals, the analogue to summing incomes in increasing order is to sum y values in decreasing order. The conventional Lorenz curve is accordingly modified as follows:

$$RL(j, b\iota_n - Y) := \frac{1}{n(b - \bar{y})} \sum_{i=1}^{j} (b - y_{[i]}), \qquad j = 1, \ldots, n \tag{6}$$

In particular if we define the new variable $z := b - y_i$, it then follows that $z_{(i)} = b - y_{[i]}$ and that $\bar{z} = b - \bar{y}$. That is,

$$\frac{1}{n(b - \bar{y})} \sum_{i=1}^{j} (b - y_{[i]}) = \frac{1}{n\bar{z}} \sum_{i=1}^{j} z_{(i)}, \qquad j = 1, \ldots, n \tag{7}$$

and the Lorenz curve $RL(j, b\iota_n - Y)$ is the classical Lorenz curve formula for the variable z.

The proposition below confirms that the Lorenz curve remains valid for investigating inequality orderings in the present context that takes into account the decreasing nature as well as the survival threshold of the health indicator. In what follows, we first state a result relating social welfare attainment and the relative Lorenz curves of two distributions with identical means. In the subsequent Proposition the comparison is extended to cover distributions with variable sum totals.

Proposition 2 *Let $X, Y \in [m, b)^n$ denote two distributions of a welfare decreasing health variable, of identical sum totals: $\sum_{i=1}^n x_i = \sum_{i=1}^n y_i$. The following statements are equivalent:*

(i) $RL(j, b\iota_n - Y) \geq RL(j, b\iota_n - X)$ for all $j = 1, \ldots, n$,

(ii) Y is obtained from X via a finite sequence of Pigou-Dalton transfers,

(iii) $\frac{1}{n}\sum_{i=1}^{n}\phi(b - y_i) \leq \frac{1}{n}\sum_{i=1}^{n}\phi(b - x_i)$ *for any convex function ϕ defined on the interval (m, b).*

In the same way it is possible to rank two distributions of a welfare decreasing health variable defined on the interval $[m, b)$ in terms of social welfare. Define the generalized Lorenz curve $GL(j, b\iota_n - Y)$ at the j-th ordinate as follows,

$$GL(j, b\iota_n - Y) := \frac{1}{n}\sum_{i=1}^{j}(b - y_{[i]}), \qquad j = 1, \ldots, n \qquad (8)$$

In the same way, if $z_i := b - y_i$, the generalized Lorenz curve $GL(j, b\iota_n - Y)$ is the classical generalized Lorenz curve formula for the variable z.

The following result–adapted here in the context of welfare decreasing health variables– has been obtained by Shorrocks (2009) in the context of the ordering of distributions of unemployment duration. It is the analogue of Proposition 2 in the context of distributions of variable sum totals.

Proposition 3 *Let $X, Y \in [m, b)^n$ denote two distributions of a welfare decreasing health variable. The following statements are equivalent:*

(i) $GL(j, b\iota_n - Y) \geq GL(j, b\iota_n - X)$ for all $j = 1, \ldots, n$,

(ii) $W(b\iota_n - Y; m) \geq W(b\iota_n - X; m)$ for any social welfare function W that satisfies MON and EQUAL,

(iii) Y is obtained from X via a finite sequence of Pigou-Dalton transfers, and or decrements.

3 The effect of the upper survival threshold

The scale invariance axiom guarantees that changing the units of measurement of y, and the two thresholds m and b does not result in any change in inequality or the Lorenz curve (6) introduced in this paper. A separate concern however may have to do with disagreement about the level of the thresholds m and b. The threshold m serves to select observations in our sample. Changing its value will result in a different sample (dropping or adding observations for individuals in good health). It is more challenging however to deal with changes in the upper threshold.

Changing the upper threshold b is of course equivalent to adding an identical amount λ to each person's endowment, that is translating the distribution of resources Y to obtain a new distribution $Y + \lambda\iota_n$. It is clear that such translational shifts in the distribution of resources will usually modify the level of inequality, and also result in a shift in the relative Lorenz curve. Consider for instance the coefficient of variation $\rho(Y) := \sigma(Y)/\bar{y}$ defined as the ratio of the standard deviation to the mean. We then can easily observe that the standard deviation is translation invariant, and hence that

$$\rho(Y + \lambda\iota_n) = \frac{\sigma(Y)}{\bar{y} + \lambda} \tag{9}$$

It follows therefore that when λ is chosen to be positive, the coefficient of variation falls as a result of a translational shift. In equivalent terms, a reduction in the upper survival threshold results in a reduction of the coefficient of variation. Specifically, we are interested in evaluating the effect of change in b on the relative and generalized Lorenz forms we have introduced in the paper.

Differentiating (8) we note that the derivative of the generalized Lorenz curve with respect to b is a constant vector function that is independent of the data [2]. More simply, the generalized Lorenz curve is a linear vector function in b . This however is not the case with the relative Lorenz curve: as the vector derivative of (6) with respect to b is a function of the data Y the ordering of distributions by the relative Lorenz curve is sensitive to the choice of b. Following Kolm (1976) and Moyes (1987), it is however possible to work with inequality indices and Lorenz curves that are invariant to changes in the upper threshold b. As we shall see below, there is however a price to pay, in that the scale invariance property will have to be replaced by a translation invariance axiom.

[2] That is this gradient vector is of the form $(1/n, 2/n, \ldots, n/n)$.

3.1 The Kolm family of absolute indices and the related Lorenz curve

The key to deriving indices that are robust to changes in the upper threshold b is to replace the scale invariance axiom $SCALINV$ by a translation invariance axiom. Specifically, together the axioms ADD, $ANON$, REP, MON, $EQUAL$ and $TRANSINV$ restrict the choice of $\phi(b-y)$ to the family $v_\kappa(b-y)$ of exponential functions (Kolm, 1976a,b):

$$v_\kappa(b-y) = 1 - \exp\left(-\kappa(b-y)\right) \qquad \kappa > 0 \tag{10}$$

Accordingly, the family of social welfare functions that satisfies the above six axioms is of the form

$$W_\kappa(b-y_1, \cdots, b-y_n; m) = \frac{1}{n}\sum_{i=1}^{n} v_\kappa(b-y_i) \tag{11}$$

The equally distributed equivalent value $\hat{y}_A$ pertaining to the above family of social welfare functions satisfies the identity $v_\kappa(b-\hat{y}_A) = \frac{1}{n}\sum\limits_{i=1}^{n} v_\kappa(b-y_i)$. Specifically,

$$\hat{y}_A = b + \frac{1}{\kappa}\ln\left(\frac{1}{n}\sum_{i=1}^{n}\exp\left(-\kappa(b-y_i)\right)\right) \tag{12}$$

By analogy with Wagstaff (2002), $\hat{y}_A$ is the achievement index pertaining to the Kolm family of social welfare functions. Accordingly, we refer to $\hat{y}_A$ as the *absolute achievement index*.

It is to be noted that $\hat{y}_A$ is invariant to changes in the parameter b . Furthermore, from Proposition 1, the equally distributed equivalent income $\hat{y}_A$ will also be decreasing in Pigou-Dalton transfers, since the axioms MON and $EQUAL$ are satisfied in a relation to the Kolm family of social welfare function.

The Kolm absolute inequality index pertaining to welfare decreasing health variables accordingly is of the form:

$$\Delta_A(b\iota_n - Y; m) := \hat{y}_A - \bar{y} \tag{13}$$

Following Moyes (1987), the Lorenz curve concept that is invariant to translational shifts of the distribution (i.e. to choices of the upper threshold b) is the absolute Lorenz curve. In the context of welfare decreasing variables, this takes the form

$$AL(j, b\iota_n - Y) := \frac{1}{n}\sum_{i=1}^{j}(\bar{y} - y_{[i]}), \qquad j = 1, \ldots, n \tag{14}$$

Again, to understand how this formula is obtained, it is easiest once again to perform a change of variable and to define $z := b - y_i$. It then follows that $z_{(i)} = b - y_{[i]}$, $z_{(i)} - \bar{z} = \bar{y} - y_{[i]}$ and that $\frac{1}{n}\sum_{i=1}^{j}(z_{(i)} - \bar{z}) = \frac{1}{n}\sum_{i=1}^{j}(\bar{y} - y_{[i]})$. The absolute Lorenz curve (14) of the welfare decreasing variable y is the Moyes (1987) absolute Lorenz curve, applied to the variable z .

We summarize the above discussion with the following Proposition, that is a corollary to Moyes (1987):

Proposition 4 *Let $X, Y \in [m, b)^n$ denote two distributions of a welfare decreasing health variable. The following statements are equivalent:*

(i) $AL(j, b\iota_n - Y) \geq AL(j, b\iota_n - X)$ for all $j = 1, \ldots, n$.

(ii) $\Delta_A(b\iota_n - Y; m) \leq \Delta_A(b\iota_n - X; m)$ for all $\kappa > 0$ and for any admissible value of the upper threshold b .

4 An illustrative application: Health achievement and inequality in five Arab countries

The purpose of this section is to illustrate using data the above methodology. Specifically, we calculate in Section 4.1 AKS relative inequality and achievement indices for welfare decreasing health variables and the related Lorenz and generalized Lorenz curves. As the relative indices and Lorenz curves are sensitive to the value assigned to the upper survival threshold b, in Section 4.2 we calculate the absolute Kolm inequality and achievement indices as well as the related absolute Lorenz curves.

Our illustrative application makes use of anthropometric data on adult (non-pregnant) women of reproductive age (15 to 49). The analysis is performed using data from the latest available Demographic and Health Surveys (DHS) conducted in five Arab countries: Egypt (2015), Yemen (2013), Jordan (2012), Comoros (2012) and Morocco (2004). The anthropometric indicator of interest here is taken as body mass (BMI), calculated by the authors as the weight in kilograms divided by the square of height measured in meters. The implementation of the above methodology necessitates that we assign values for the thresholds m and b . There is no consensus on these values in the literature, with the proposed range of the optimal value m being often between 18.5 and 24.9 while the survival threshold value for the upper bound can reach 60 (considered to be a fatal level of obesity). In line with the guidelines of the World Health Organization (2004; 2017), for the purpose of the present analysis, we set the value of b equal to 60 , while the cut-off value m is fixed at 24.90 [3]. After cleaning the data for missing and mis-

[3] That is, the WHO guidelines for non-pregnant women define an optimum interval

coded values on the variable of interest, the respective sample sizes are as follows: Egypt ($n = 5226$), Jordan ($n = 6336$), Morocco ($n = 6239$), Yemen ($n = 5669$) and Comoros ($n = 1927$).

4.1 Relative health achievement and inequality

We begin by examining the relative inequality indices $\Delta_R(b\iota_n - Y; m)$ and the related achievement indices $\hat{y}_R$ as well as the corresponding generalized Lorenz curves $GL(j, b\iota_n - Y)$ in the five countries. We report in **Table 1** calculations pertaining to inequality and achievement indices in relation to two values for the inequality aversion parameter: $\beta = 1$ and 2. Consider first the results pertaining to $\beta = 1$. Recalling that achievement (welfare) is decreasing in y, we find that the anthropometric achievement index $\hat{y}_R$ ranks social welfare as lowest in Egypt ($\hat{y}_R = 33.2$) followed by Jordan ($\hat{y}_R = 31.6$), Comoros ($\hat{y}_R = 29.7$),Yemen ($\hat{y}_R = 29.5$), while it is highest in Morocco ($\hat{y}_R = 29.1$). Increasing the social aversion to inequality ($\beta = 2$) , results in lower health achievement (that is, higher values) in all countries. We note nonetheless that this does not change the ranking order of the countries.

Turning now to inequality, we find that the AKS index for welfare decreasing variables**,** $\Delta_R(b\iota_n - Y; m)$, for $\beta = 1$, takes the highest value in Egypt: 2.9%.In contrast, this figure is the lowest in Morocco 0.8%, while inequality is between these two values in the context of the other three countries. Because the mean of the distribution is highest in Egypt (32.4) and lowest in Morocco (28.8), health achievement is lowest in Egypt because of a higher mean and dispersion of the underlying dispersion. Once again, the magnitudes of inequalities increase with the inequality aversion parameter. For instance, for $\beta = 2$, inequality in Egypt, is the highest at approximately 8.3%. This is followed by 4.5% in Jordan, 3.0% in the Comoros and 2.5% in Yemen, while it is the lowest in Morocco (about 1.8%).

To investigate systematically the welfare ordering of these five countries the generalized Lorenz curves pertaining to the BMI distributions are plotted in **Figure 1**. The generalized Lorenz curve of a hypothetical optimum health distribution $Y^* = (m, \ldots, m)$, would take the form of a straight line starting at zero with a slope equal to $b - m$. This would entail the mean of the distribution $\bar{y}$ approaching $m = 24.90$, the anthropometric achievement index, $\hat{y}_R$, approaching $\bar{y}$ and the inequality index, $\Delta_R(b\iota_n - Y; m)$, approaching zero. We can view the process of improvement in the distribution of body mass as one where the generalized

with values of m ranging between 18.5 and 24.9. Likewise, the fatal range involves any body mass value in excess of 60. Our chosen value of m is therefore the upper bound of the optimum range, while b is set at the lower bound of the fatal range.

Lorenz curves approach from below the generalized Lorenz curve of the optimum distribution. In accordance with Proposition 3, the generalized Lorenz curve of Egypt lying below the other four curves, (see **Figure 1**) and that of Morocco being closest to the straight line, entails that the above welfare ordering of the five countries is robust to the choice of value assigned to the inequality aversion parameter β.

As argued in Section 3 nonetheless, these findings are not necessarily robust to choice of the survival threshold b. For instance, increasing the survival threshold to $b = 65$, and setting the inequality aversion parameter to $\beta = 1$, the value of inequality for Egypt decreases to 1.9% (compared with 2.9%) while that for Morocco falls to 0.6% (compared with 0.8%) (see second panel of **Table 1**). We also illustrate the relative Lorenz curves pertaining to Egypt and Morocco in relation to the two survival thresholds $b = 60$ and $b = 65$. As shown in **Figure 2**, we observe that the curvature of the relative Lorenz curve is altered by changes in b.

As the results discussed above are sensitive to the choice of b, in the following subsection we report findings pertaining to the Kolm family of absolute inequality and achievement indices and related absolute Lorenz curve.

4.2 Absolute health achievement and inequality

To explore systematically an inequality ordering of countries that is robust to the choice of upper threshold values, we depict in **Figure 3** absolute Lorenz curves for the five countries of interest. To read these findings, we can observe that the perfect equality line of the absolute Lorenz curve coincides with the horizontal axis. In our context, this line indicates a distribution where everyone has the same body mass value. The further an absolute Lorenz curve dips from the perfect inequality line, the higher the level of inequality. As the five absolute Lorenz curves do not intersect, we conclude, in accordance with Proposition 4, that absolute inequality is lowest in Morocco and highest in Egypt. The result is of interest, as it reveals that for all inequality aversion parameter $\kappa > 0$ absolute inequality indices for welfare decreasing variables (13) will order the countries in the same way as absolute Lorenz curve criterion.

Computations of alternative absolute achievement, $\hat{y}_A$, and inequality indices, $\Delta_A(b\iota_n - Y; m)$, are reported in **Table 2**. We discuss briefly the findings related to $\kappa = 1$. In this regard, it is to be noted that the welfare ranking of the countries, as measured by the absolute achievement and inequality indices remains the same as the one observed using the relative indices. Health achievement remains the lowest in Egypt ($\hat{y}_A = 51.7$) and the highest in Morocco ($\hat{y}_A = 44.8$). Similarly, inequality remains highest in Egypt ($\Delta_A = 19.3$) and lowest in Morocco ($\Delta_A = 16.0$). Interestingly, the values of these absolute indices are invariant to the choice of the survival threshold.

5 Conclusion

The last two decades have placed the measurement of inequality and achievement in health at the center of the development debate. The purpose of our paper was to address the question of the measurement of pure health inequalities and achievement in the context of welfare decreasing variables. We were thus led to adopt a general framework whereby the health variable is reported on an interval, from an optimum level m to a critical level b, beyond which survival was no longer assured.

We have noted in our discussion above that the context of inequality measurement *per se* on a welfare decreasing variable was not the problem. Specifically, we have argued that as the utility function was Schur-concave (be it increasing or decreasing in the underlying health indicator) and the associated inequality index was Schur-convex, the Lorenz curve could be used to order health distributions in the same way that it was applied to order income distributions.

There were however two significant problems that required some departures from the usual framework used to measure inequality and social welfare in relation to income distributions. Firstly, we have shown in Proposition 1 that for welfare decreasing variables, the equally distributed equivalent value - the summary statistic used to derive health achievement indices - is a Schur-convex function: that is, a function that is decreasing in progressive transfers. This is the opposite of what we should expect of such a summary statistic. This has meant that the relative Atkinson-Kolm-Sen inequality indices available from the income inequality literature required some adaptation in the context of welfare decreasing variables. Accordingly, appropriate achievement and inequality indices for welfare decreasing variables were introduced in Sections 2 of the paper.

The second problem that required attention was that of survival thresholds, a property inherent to many health indicators. We have acknowledged that clinical research informs about sensible values of the survival threshold b. Nonetheless, it remained that the Lorenz curve and the associated scale invariant inequality indices were not robust to alternative values of the survival threshold. For this second problem we have argued that the family of translation invariant social welfare functions introduced by Kolm (1976a,b) and related absolute Lorenz curve (Moyes, 1987) allowed us to undertake inequality comparisons between distributions that are robust to the chosen level of the survival threshold.

Translation invariant achievement and inequality indices in the context of welfare decreasing variables, were accordingly introduced in Section 3 and an illustrative application of the methodology was provided in Section 4 of the paper.

One important extension of our framework would consist in deriving achievement and inequality indices of welfare decreasing variables in the context of the socio-economic disparities in health. This would complement the readily available normative framework existing in this literature in relation to welfare increasing health indicators (Wagstaff, 2002; Erreygers, 2013). Another possible extension of the analysis could consist in remaining centered in the context of disparities in pure health, but adopting a multidimensional perspective on the measurement of achievement and inequality, where the health variable is welfare decreasing.

6 Appendix

Proof of Proposition 1 Because, by assumption, $u(.)$ is a strictly decreasing and differentiable function, it follows, that u^{-1} exists, is strictly decreasing and differentiable on the interval $[u(b^o - b)\ ,\ u(m^o - b)]$. Let $t := \frac{1}{n}\sum_{i=1}^{n} u(b - y_i)$ be an element of the interval $[u(b^o - b)\ ,\ u(m^o - b)]$, and define the function $h(y_1, \ldots, y_n) := u^{-1}\left(\frac{1}{n}\sum_{i=1}^{n} u(b - y_i)\right) = u^{-1}(t)$, so that $\hat{y} = h(y_1, \ldots, y_n)$.

Our next task is to show that $h(y_1, \ldots, y_n)$ is a Schur-convex function. On the basis of-Remark 3.A.5 of Marshall et al (2011 p. 85), in showing this, without loss of generality we may readily consider the case of $n = 2$ individuals with endowments $y_i > y_j$. Because $u(.)$ is concave, the social welfare function W is Schur-concave and

$$(y_i - y_j)\left(\frac{\partial u}{\partial y_i} - \frac{\partial u}{\partial y_j}\right) \leq 0$$

On the other hand,

$$(y_i - y_j)\left(\frac{\partial h}{\partial y_i} - \frac{\partial h}{\partial y_j}\right) = (y_i - y_j)\frac{\partial u^{-1}}{\partial t}\left(\frac{\partial u}{\partial y_i} - \frac{\partial u}{\partial y_j}\right) \geq 0 \ ,$$

since the inverse function $u^{-1}(.)$ is strictly decreasing and therefore $\partial u^{-1}/\partial t < 0$. It follows therefore that $\hat{y} = h(y_1, \ldots, y_n)$ is a Schur convex function, that is a function decreasing in Pigou-Dalton transfers.

Because $u()$ is decreasing, we have furthermore that $u(b - \bar{y}) \geq u(b - \hat{y}) \iff \hat{y} \geq \bar{y}$. $\square$

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Tables

Table 1: Health Achievement and Inequality in Five Arab Countries						
(m=24,9; b=60)						
β	Countries	Egypt	Jordan	Comoros	Yemen	Morocco
	Sample size	5226	6336	1927	5669	6239
	Mean	32,3984	31,0187	29,3406	29,2210	28,8199
1	Achievement index	33,2012	31,5582	29,7116	29,5411	29,0668
	AKS inequality index	0,0291	0,0186	0,0121	0,0104	0,0079
2	Achievement index	34,6960	32,3165	30,2569	29,9890	29,3743
	AKS inequality index	0,0832	0,0448	0,0299	0,0250	0,0178
(m=24,9; b=65)						
β	Achievement index	33,0173	31,4532	29,6405	29,4824	29,0256
1	AKS inequality index	0,0190	0,0128	0,0084	0,0073	0,0057

Table 2: Absolute Health Achievement and Inequality in Five Arab Countries						
(m=24,9; b=60 & b=65)						
κ	Countries	Egypt	Jordan	Comoros	Morocco	Yemen
	Sample size	5226	6336	1927	6239	5669
	Mean	32,3984	31,0187	29,3406	28,8199	29,2210
0,5	Achievement index	45,8704	42,9668	41,7929	37,7921	40,9357
	Kolm inequality index	13,4719	11,9480	12,4523	8,9722	11,7147
1	Achievement index	51,6937	49,1526	47,7472	44,8288	47,6321
	Kolm inequality index	19,2952	18,1338	18,4066	16,0089	18,4112

Figures

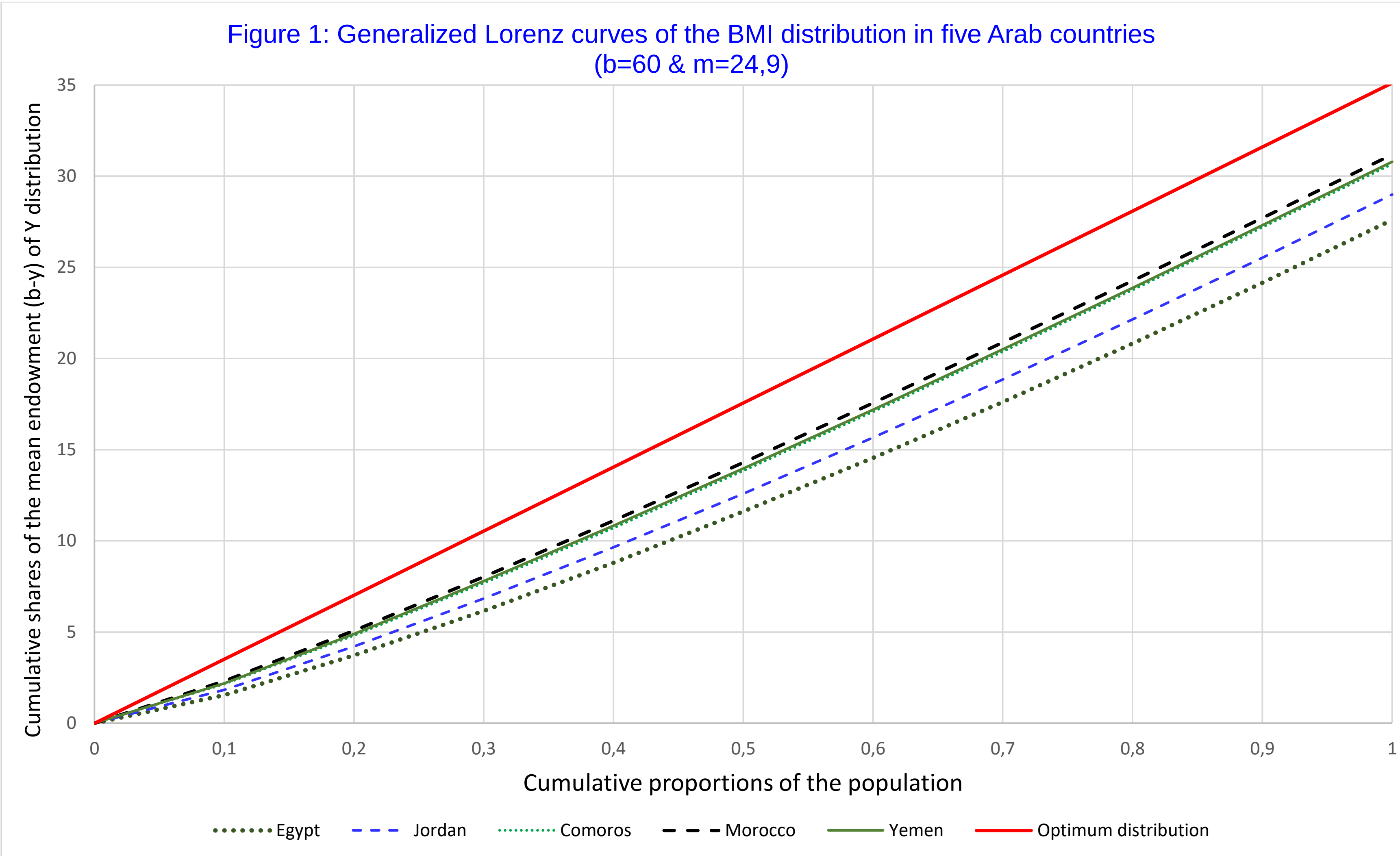
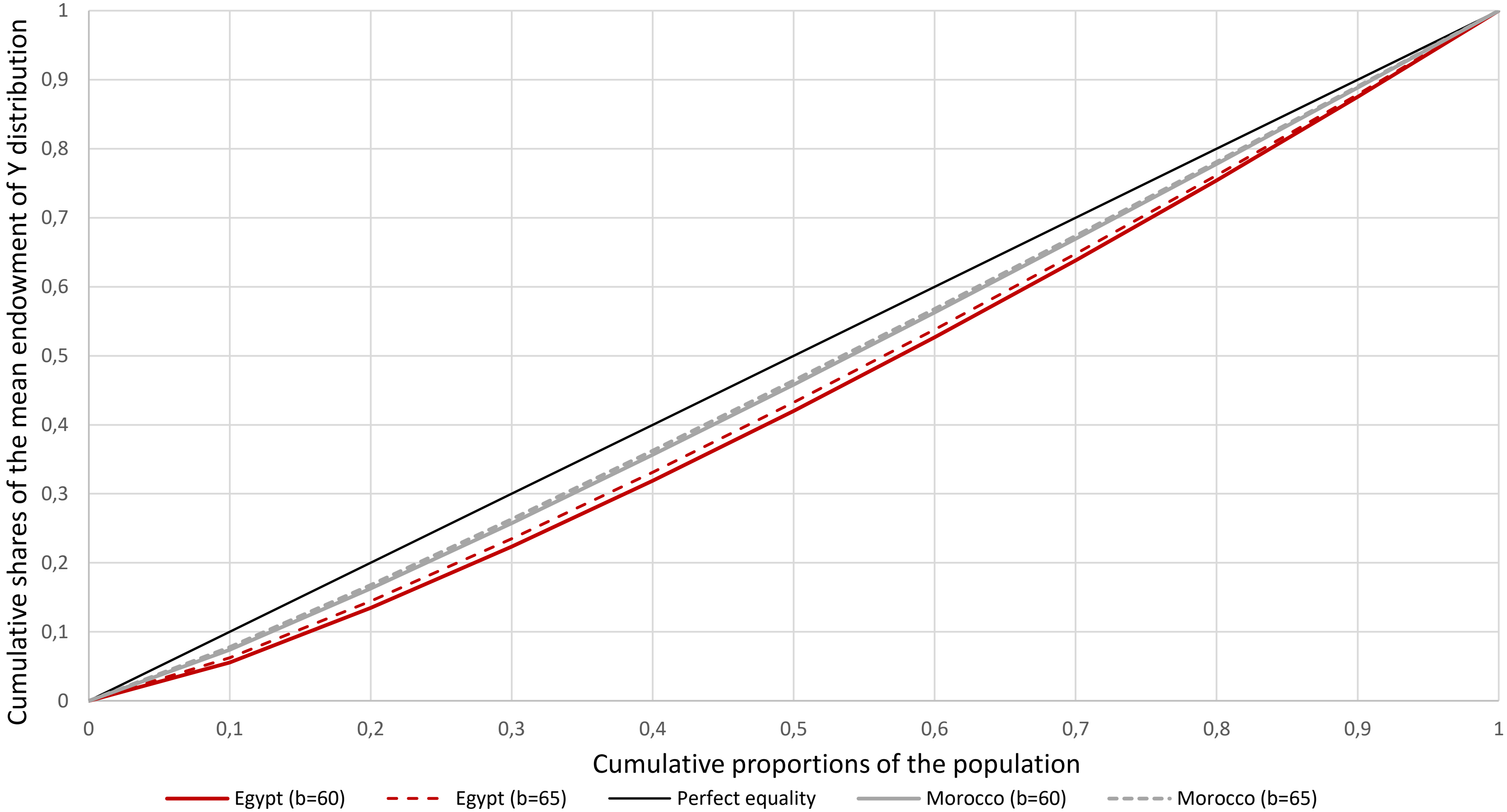


Figure 1: Generalized Lorenz curves of the BMI distribution in five Arab countries
(b=60 & m=24,9)
Cumulative shares of the mean endowment (b-y) of Y distribution
35
30
25
20
15
10
5
0
0
0,1
0,2
0,3
0,4
0,5
0,6
0,7
0,8
0,9
1
Cumulative proportions of the population
Egypt
Jordan
Comoros
Morocco
Yemen
Optimum distribution

Figure 2: Relative Lorenz curves of the BMI distribution of Egypt and Morocco for different values of the survival threshold (b)



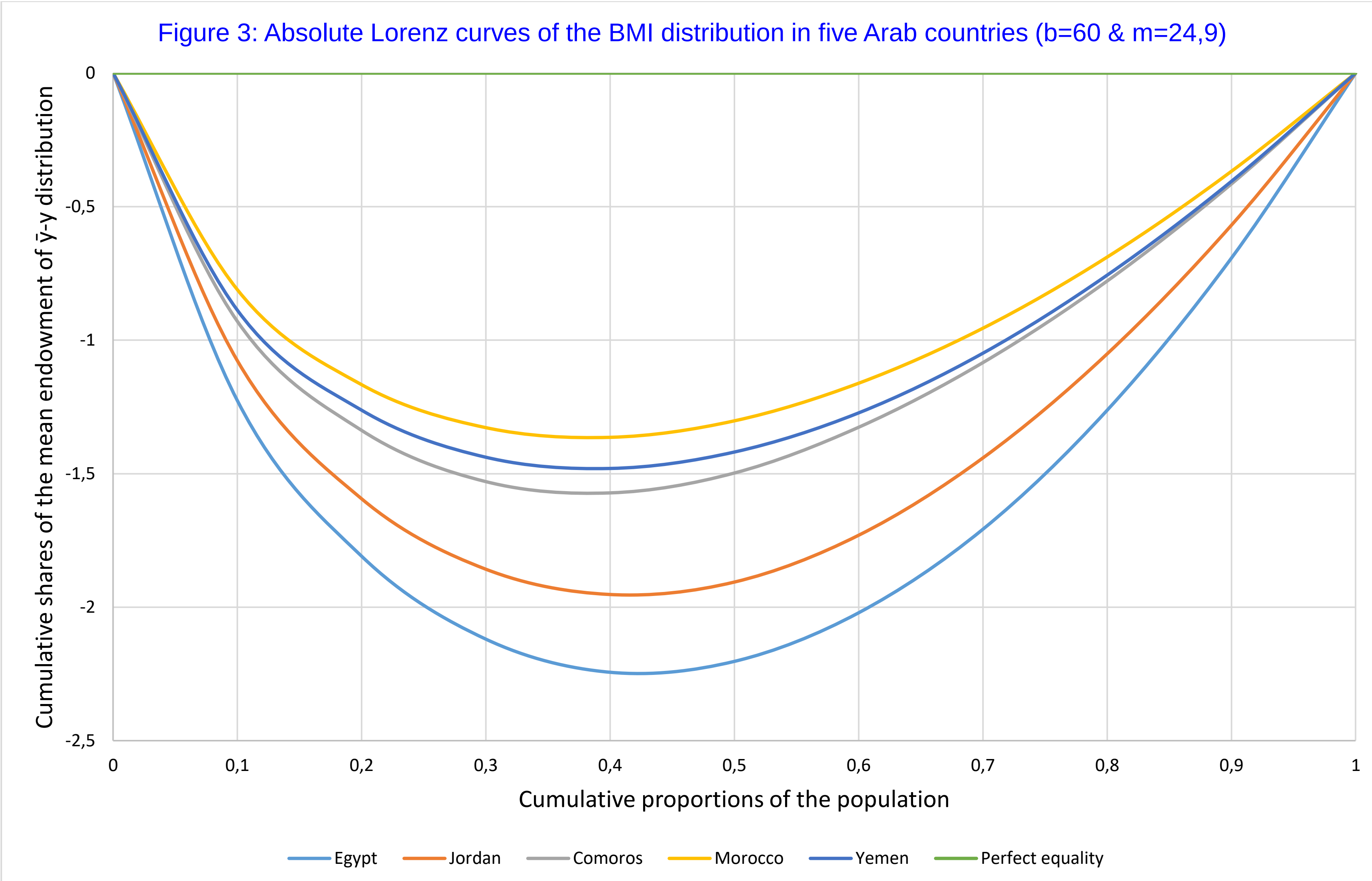


Figure 3: Absolute Lorenz curves of the BMI distribution in five Arab countries (b=60 & m=24,9)
Cumulative shares of the mean endowment of ȳ-y distribution
0
-0,5
-1
-1,5
-2
-2,5
0
0,1
0,2
0,3
0,4
0,5
0,6
0,7
0,8
0,9
1
Cumulative proportions of the population
Egypt
Jordan
Comoros
Morocco
Yemen
Perfect equality